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Black hole entropy, topological entropy and the Baum–Connes conjecture in K-theory

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Abstract
We shall try to show a relation between *black hole* (BH) entropy and *topological entropy* using the famous *Baum–Connes conjecture* for *foliated manifolds* which are particular examples of *noncommutative spaces*. Our argument is *qualitative* and it is based on the microscopic origin of the Beckenstein–Hawking area–entropy formula for BHs, provided by superstring theory, in the more general noncommutative geometric context of M-theory following the approach of Connes–Douglas–Schwarz.

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This article is dedicated to Stavroula Giannoutsou.

1. Introduction and motivation

We know from a series of articles back in 1996 (Strominger, Vafa, Maldacena and Horowitz [12]) that superstring theory can in some cases (*multicharged extremal black holes* (BHs) and for large values of charges) give an explanation for the *microscopic origin* of the quantum states associated with a BH, which give rise to its quantum mechanical entropy described by the Beckenstein–Hawking area–entropy formula.

The argument relies heavily on *S-duality* which provides a way to *identify perturbative string states and D-branes*; these are all *BPS* states in the *weak-coupling* region, with *extremal BHs with NS and R charges*, respectively, in the *strong-coupling* region. For simplicity we assume no backcreation for the BH (that is, that the energy which is equal to its mass is *constant*) and that makes it reasonable to count only BPS states in string theory since these states have the important property that their mass does not receive any quantum correction.

A crucial detail to bear in mind is that since superstring theory lives in ten dimensions and that the Beckenstein–Hawking formula refers (originally) to four dimensions, the extra dimensions have to be *compactified*; hence compactification is important in establishing this relation. The following picture is not completely correct, but is very helpful for understanding what is actually going on: the compactified dimensions are treated like a ‘phase space’ which after quantization provides the quantum states associated with the BH, thus giving rise to its entropy.

In 1998, however, the now ‘classical’ article by Connes *et al* [9] taught us that *M-theory*, which is a *generalization* of superstring theory, admits *additional* compactifications on *noncommutative spaces*, in particular *noncommutative tori*.

Then the natural question is: What would happen if in the scenario considered by Strominger, Vafa *et al*, we now assume that the *compactified dimensions* form a *noncommutative space*? We shall try to give a *qualitative answer* to the above question mainly based on *(noncommutative) topology*. Before doing that, we shall make some brief remarks on both M-theory and noncommutative geometry.

We start with M-theory: until the mid-1990s we had five consistent superstring theories: types I, IIA, IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. Following the discoveries of various string dualities, it is now believed that these five theories are an artifact of perturbation expansion: there is only *one fundamental eleven-dimensional theory* called *M-theory* which contains p-dimensional extended objects called *p-branes*. For example, point particles are 0-branes, strings are 1-branes etc. Rather few things are known about this underlying theory and the basic strategy is to try to understand this M-theory from its *limiting theories* which are the five superstring theories in ten dimensions and eleven-dimensional supergravity.

Next we shall try to give an idea of what *noncommutative geometry* is. The motivation for the development of this *new branch of mathematics* is actually twofold:

(1) Descartes introduced *coordinates* in the 17th century and revolutionized geometry. Subsequently that gave rise to the notion of the *manifold*. One important generalization introduced by Connes (see [7]) was the notion of a *noncommutative manifold*. Roughly, one can think of a ‘generalized manifold’, or a ‘noncommutative manifold’, as a space having a corresponding coordinate function space which locally ‘looks like’ an *operator algebra*, in fact a C^*-algebra which in general is *noncommutative*. This is strongly reminiscent of quantum mechanics and sometimes these are called *‘quantum spaces’*. The origin is essentially *Gelfand’s theorem* which states that the *category* of (unital) *commutative* C^*-algebras with $*$-preserving homomorphisms is equivalent to the *category* of (compact) *locally compact Hausdorff spaces* with homeomorphisms.

(2) We would like to generalize the *index problem* solved by Atiyah and Grothendieck in the late 1960s. The origin came from *Quillen’s higher algebraic K-theory*, a *simplification* of which is the *K-theory of* (not necessarily commutative) *C^*-algebras* which we shall use later. Then *Serre–Swan theorem* identifies it with Atiyah’s original K-theory in the *commutative case* using Gelfand’s theorem.

We think that the idea behind the first motivation is quite clear and in fact *this* idea is behind the vast majority of articles in the physics literature up to now which make some use of noncommutative geometry. We shall not give the precise definitions here. The interested reader may study [7] which also contains an exhaustive list of references on the subject.

However, in this paper we would like to elaborate more on the ideas behind the *second motivation*, namely the *index theory;* in fact one of the aims of this present paper is to try to make some use of the ideas behind it in physics and we shall start by explaining what *index theory* is (we have been influenced in our presentation by [10] which is an excellent article).

Index theory is an attempt to unify *topology* and *analysis*. The formal way to do that is to manufacture two mathematical objects (two *K-theories*), one containing the topological data and the other containing the analytical data, and then we compare them; more concretely, given a ‘commutative’ space M (namely a manifold or an algebraic variety), one constructs *two K-theories*: one is called *topological* and contains all stable isomorphism classes of (say) complex vector bundles over the space M. The other is called *analytical* but we shall adopt the more recent term *K-homology* and contains all homotopy classes of principal symbols of elliptic pseudodifferential operators acting on M (more precisely on sections of vector bundles over M). What we describe is Atiyah's *Ell* group from which K-homology evolved subsequently.

Grothendieck proved that for any commutative space the analytical and the topological K-theories are *isomorphic* and thus one can say that essentially the *Atiyah–Singer index theorem* gives the *explicit isomorphism*.

One also has two natural maps from these two K-theories to the integers: for the *topological K-theory* it is given by the *Chern character* and for *K-homology* it is given by the *(Fredholm) index* of the operator. Then the Atiyah–Singer index theorem says that the index differs from the Chern character essentially by the *Todd class*.

Remark 1. The relation between topology and analysis is quite deep; the Atiyah–Singer index theorem gives a relation between *primary* invariants (Chern classes and index). There are also relations between *secondary invariants*, which are more *delicate objects* like *Chern–Simons* forms for bundles and Atiyah's intriguing *η-invariant* for operators (related to Riemann's famous 'zeta' function). The *Jones–Witten topological quantum field theory* for 3-manifolds is such an example; if one thinks of it as the *non-Abelian version* of Schwarz's original work where he observed that there is a close relation between the partition function of Abelian Chern–Simons 3-form (degenerate quadratic functionals) and the *Ray–Singer analytic torsion* (the η-invariant of the Laplacian) which is a topological invariant of the 3-manifold considered (see [1]).

Remark 2. Each of the above two K-theories essentially consists of two Abelian groups due to Bott periodicity, namely we have topological $K^0(M)$ and $K^1(M)$ and analytical $K_0(M)$ and $K_1(M)$, where in the latter we have put the indices as subscripts to indicate that this is a homology theory (K-homology).

The *Baum–Connes conjecture* then is an analogous generalized statement for *analytical* and *topological K*-theories *appropriately defined* for *noncommutative spaces*; in fact in its most general formulation it refers to categories with inverses (groupoids).

We shall only mention here that the basic tool for constructing these K-theories for categories is essentially the *Quillen–Segal* construction (see for example [3] and references therein).

2. Microscopic origin of black hole entropy

We shall treat the simplest example appearing in [12] (see moreover [18] which is a nice review article on the subject).

Consider for convenience a 5-dim (five-dimensional) BH with three charges Q_1, Q_5, n. Since superstrings require ten dimensions, we assume the remaining 5 dims are *compactified* on a fixed torus of volume $(2\pi)^4 V$ which is constant and the fifth remaining direction is another circle of circumference $2\pi R$, where this radius is much bigger than those of the other four circles in the 4-torus. One can compute using BH quantum mechanics that

$$S_{BH} = \frac{A}{4G} = 2\pi\sqrt{Q_1 Q_5 n}.$$

The same result can be obtained from string theory considerations: apart from the metric, one has an NS field H (3-form) with both electric and magnetic charges denoted by Q_1, Q_5 and n

is the quantization of the momentum $P = n/R$ along the large circle. If we assume type IIB superstring theory and start from flat 10-dim (ten-dimensional) space-time we compactify on the 5-torus as described above. The objects which carry the charges Q_1 and Q_5 turn out to be respectively a D-string wrapped Q_1 times around the big circle of radius R and a $D5$-brane wrapped Q_5 times around the 5-torus. We would like to underline here that the calculation appearing in [12] is an index theoretical one, because what the authors use in order to count BPS states is the *supersymmetric index*.

Our question, which we mentioned in the first section, was that of how this formula should be modified if we assume that the compactified 5-torus is a *noncommutative* one. In addition we shall also assume that the noncommutative 5-torus is an ordinary 5-torus which carries a *foliation structure*. The reason for this is that the *spaces of leaves* of foliations can be really 'very nasty spaces' from the topological point of view and in most cases they are not (ordinary) manifolds. So foliated manifolds are particular examples of noncommutative manifolds. More details and examples can be found in [7].

Suggestion. The difference will be in the topological charge Q_5. We should use an *invariant* for *foliated manifolds*. Our suggestion is the *new* invariant introduced in [2] coming from the pairing between *K-homology* and *cyclic cohomology*. The formula is

$$\langle [e], [\phi] \rangle = (q!)^{-1} (\phi \# \mathrm{Tr})(e, \ldots, e)$$

where $e \in K_0(C(F))$, $\phi \in HC^{2q}(C(F))$ and # is the *cup product* in cyclic cohomology introduced by Connes. In the above formula we denote by F the codim-q foliation of the 5-torus, $C(F)$ is the C^*-algebra associated with the foliation (which comes after imposing a suitable *C^*-algebra 'completion'* on the *holonomy groupoid* of the foliation) and finally $[e]$ and $[\phi]$ are *'canonical' classes* associated with the foliation. The first one is a naturally chosen *closed transversal* and the second is the *fundamental cyclic cocycle* of the foliation. Moreover, $K_0(C(F))$ and $HC^{2q}(C(F))$ denote the 0th K-homology group and the $2q$th cyclic cohomology group of the corresponding C^*-algebra of the foliation respectively. (More details and precise definitions can be found in [2].)

The definition of the above invariant uses K-homology, i.e. it is *operator algebraic*. That means that it lies in the *analytical world*. (The above framework uses the language of C^*-algebras which by definition is a combination of algebra and functional analysis.) We would like to see what it corresponds to in the topological world. This would have been very straightforward if we had known that the Baum–Connes conjecture was true.

Some years ago a *deep* theorem was proved by Gerard Duminy which refers to foliated manifolds as well but it uses *topological tools*, in particular the *Godbillon–Vey (GV) class* (which we define in the next section); hence it lies in the *topological world*. So firstly one should try to understand the relation between *our invariant* which is operator algebraic and the *GV class*. For the moment only a few things can be said [8]:

An important property of the operator algebraic invariant is that in the *commutative case*, namely for a fibre bundle, it does not vanish as the GV class does (recall that the GV class is a particular class in the Gelfand–Fuchs cohomology) but reduces to the usual characteristic classes (a linear combination of the Chern class of the bundle which is the foliation itself, plus the Pontrjagin class of the tangent bundle of the base manifold which in this case is the normal bundle of the foliation, see [2]).

Based on the above commutative example, a *qualitative picture* is that in the general case of an arbitrary foliation, this invariant has contributions from *two parts:* the first is some *Chern (or Pontrjagin) class* of the *normal bundle* of our foliation and the second is some *characteristic* *class* of our *foliation* itself, namely a class of the corresponding Gelfand–Fuchs cohomology. Moreover, we know from the Duminy theorem that (for codim-1 cases) the GV class is related to the topological entropy and thus the second, noncommutative part of our invariant should 'contain' the difference in the entropy.

Put differently, what we are trying to do is to understand some of the mysteries of the Baum–Connes conjecture in the particular case of foliated manifolds (*differential topology versus operator algebras*). We have not succeeded in doing this, but we think it is worth reviewing the topological side of the story along with Duminy's theorem. Needless to say, the Baum–Connes conjecture is one of the major mathematical problems still open today which attracts a lot of interest from pure mathematicians.

3. Duminy's theorem

To a large extent, what we know for the topology of foliated manifolds is essentially due to the pioneering work of Thurston in the late 1970s and it refers primarily to codim-1 foliations on closed 3-manifolds.

There is only one known invariant for foliated manifolds, which is roughly the analogue of the Chern classes for bundles: this is the celebrated *GV class* which belongs to the *Gelfand–Fuchs cohomology*.

Let us review some basic facts for *foliated manifolds;* roughly they generalize fibre bundles (the total space of every fibre bundle is a foliation, the fibres are the leaves):

By definition a codim-q *foliation* F on an m-manifold M is given by a codim-q *integrable sub-bundle* F of the tangent bundle TM of M. 'Integrable' means that the Lie bracket of vector fields of F closes. This is the *global* definition of a foliation.

There is an equivalent *local* definition: a codim-1 foliation F on a smooth m-manifold M can be defined by a non-singular 1-form ω vanishing exactly at vectors tangent to the leaves. Integrability of the corresponding $(m-1)$-plane bundle F of TM implies that $\omega \wedge \mathrm{d}\omega = 0$ or equivalently $\mathrm{d}\omega = \omega \wedge \theta$ where θ is another 1-form. The 3-form $\theta \wedge \mathrm{d}\theta$ is closed and hence determines a de Rham cohomology class called the *GV* class of F.

Although ω is only determined by F up to multiplication by nowhere-vanishing functions and θ is determined by ω only up to addition of a d-exact form, the GV class actually depends only on the foliation F. The GV class can also be defined for foliations of codim $q \geqslant 1$: in this case one needs a decomposable non-singular q-form ω and then the integrability condition is as above, $\omega \wedge \mathrm{d}\omega = 0$, or equivalently $\mathrm{d}\omega = \omega \wedge \theta$, where θ is another 1-form. The $(2q+1)$-form $\theta \wedge (\mathrm{d}\theta)^q$ is closed and hence determines a de Rham cohomology class which is the GV class for our codim-q foliation.

Note that following the global definition of a foliation given above, the sub-bundle F of the tangent bundle TM of M is itself an honest bundle over M and thus it has its own characteristic classes from Chern–Weil theory. This theory however is *unable* to detect the *integrability property* of F and for this reason we had to develop the Gelfand–Fuchs cohomology, a member of which is the GV class.

The key thing to understand about foliations is that a codim-q foliation F on an m-manifold M gives a *decomposition* of M into a *disjoint union* of submanifolds called *leaves* all of which have the same dimension $(m-q)$. The definition of a foliation seems rather 'innocent', at least the global one, possibly because it is very brief. Yet this is very far from being true. One has *two fundamental differences* between a foliation and the *total space* of a fibre bundle:

(1) The leaves of a foliation in general have *different fundamental groups* whereas for a bundle the fibres are the *'same'* (homeomorphic, diffeomorphic or homotopy equivalent) as some fixed space called the *typical fibre*. Thus generically one has no control on the homotopy types of the leaves; under some very special assumptions however (e.g. restrictions on the

homology groups of the manifold which carries the foliation) one may get 'some' control on the homotopy types of the leaves and in these cases we obtain some deep and powerful theorems, the so-called *stability theorems*; one such is Thurston's stability theorem (see [13] or [5] for example).

The above fact, along with the *holonomy groupoid* of the foliation (roughly the analogue of the *group of gauge transformations* for principal bundles) gives rise to a corresponding *noncommutative algebra* which one can naturally associate with any foliation using a construction due to Connes; for fibrations the corresponding algebra is essentially *commutative*. ('Essentially' means it is *Morita equivalent* to a commutative one; for the proof see [2].) Moreover, some leaves may be compact and some others may not be.

(2) The leaves are in general *immersed* submanifolds and *not embedded* like the fibres of a fibration. In both cases normally there is no intersection among different leaves and fibres (we assume for simplicity no singularities), so in both cases one can say that we have a notion of *parallelism*. For foliations it is far more general; that can give rise to *topological entropy*. This notion was introduced by topologists (Ghys, Langevin and Walczak) in 1988 (see [11] or [5]).

We need one further definition before we state Duminy's theorem: a leaf L of a codim-1 foliation F is called *resilient* if there exists a transverse arc $J = [x, y)$ where $x \in L$ and a loop s on L based on x such that $h_s : [x, y) \to [x, y)$ is a contraction to x and the intersection of L and (x, y) is non-empty. (Note that in the definition above the arc J is *transverse* to the foliation.) Intuitively a resilient leaf is one that 'captures itself by a holonomy contraction'. The terminology comes from the French word *'ressort'* which means *'spring-like'*. We are now ready to state:

Duminy's theorem. *'For a codim-1 foliation F on a closed smooth m-manifold M, one has that $GV(F) = 0$ unless F has some (at least one) resilient* leaves'.

The *proof* is very long and complicated and it uses a theory called *architecture of foliations* (see [5] and [6]). The important lesson from Duminy is that for topology, *only resilient leaves matter*, since only they contribute to the GV class.

As a very interesting corollary of the above theorem we get the relation between the GV class and *topological entropy*. To define this notion one has first to define the notion of *entropy of maps* and then generalize it for foliations using as intermediate steps the entropy of transformation groups and pseudogroups.

In general, *entropy measures the rate of creation of information*. Roughly, if the states of a system are described by iteration of a map, states that may be *indistinguishable* at some initial time may diverge into clearly *different* states as time passes. Entropy measures the *rate* of creation of states. In the mathematical language, it measures the *rate of divergence of orbits of a map*.

We shall give a qualitative description: let f be a map from a compact manifold onto itself. To measure the number of orbits one takes an empirical approach, not distinguishing ε-close points for a given $\varepsilon > 0$. If x and y are two indistinguishable points, then their orbits $\{f^k(x)\}_{k=1}^{\infty}$ and $\{f^k(y)\}_{k=1}^{\infty}$ will be distinguishable provided that for some k, the points $f^k(x)$ and $f^k(y)$ are at distance greater than ε. Then one counts the number of distinguishable orbit segments of length n for fixed magnitude ε and looks at the growth rate of this function of n. Finally one improves the resolution arbitrarily well by letting $\varepsilon \to 0$. The value obtained is called *the entropy of f* and it *measures* the *asymptotic growth rate of the number of orbits of finite length as the length goes to infinity*.

The above can be rigorously formulated and one can define the *entropy of a foliation* to be a *non-negative real number* (see [11]).

One then can prove:

Proposition 1. *If the compact foliated space (M, F) has a resilient leaf, then F has positive entropy.*

The proof can be found in [5].

Combining this with Duminy's theorem (for *codim-1 case*) we get the following:

Corollary 1. *If (M, F) is a compact (C^2-)foliated manifold of codim-1, then zero entropy implies $GV(F) = 0$.*

The converse is *not* true; one can see counter-examples in [5].

4. Physical discussion

Topologically, the difference between the commutative charge and the noncommutative one is the *topological entropy* of the foliated torus. Commutative spaces can be considered to have *zero* topological entropy, whereas foliations *may* have non-zero topological entropy. For example *the topological entropy of fibre bundles is zero* and hence from the corollary to Duminy's theorem above, so is the GV class for fibre bundles. More generally the GV class vanishes for any foliation defined by *closed forms* but the topological entropy vanishes for foliations defined by closed forms only in the codim-1 case. To see that fibre bundles are particular examples of foliations defined by closed forms, note that the (closed) form defining the fibre bundle is the pull-back of the volume form on the base manifold and this is obviously closed. (We thank A Candel and L Conlon for pointing this out to us.)

Note. *Not every noncommutative space has non-zero topological entropy*. Duminy's theorem tells us that this is 'captured' by the GV class.

Physically, one can try to think of some 'critical point' where the foliation becomes 'wild enough' to develop resilient leaves, thus have non-zero GV class and thus non-zero topological entropy. Geometrically, the parameter which indicates the transition from the commutative to the noncommutative realm is exactly the GV class, since it is the parameter which signifies the appearance of non-zero topological entropy. It would be very interesting to try to see whether the GV class has any direct physical meaning: one suggestion would be that it might be related to the *curvature of the B-field* for the *codim-1 case* in some appropriate context (see [4] for more details).

Moreover, it is very desirable from the physical point of view to try to find a *quantitative* description of this scenario via a *direct* computation using *(almost) BPS* states. Some recent work (mainly in the last year) due to Konechny and Schwarz [14] might be useful in this direction. Let us fix our notation: T^d denotes the *commutative* d-torus and T^d_θ denotes the *noncommutative* d-torus. Of particular interest is the case of *noncommutative* $\mathbb{Z}_2$ and $\mathbb{Z}_4$ *toroidal orbifolds* considered by Konechny–Schwarz in their most recent articles [14].

The role of supersymmetry is very important: our understanding is that supersymmetry prevents the foliation from becoming very messy. Supersymmetry and topological entropy are mutually *'competing'* notions. We would like to find *how much supersymmetry has to be preserved to ensure that the topological entropy remains zero*.

For example, in all the cases considered in the Connes–Douglas–Schwarz article [9], the foliations of the tori were *linear* (*Kronecker foliations* as they are known in geometry), so topologically they were spaces with zero topological entropy. That was dictated by their *maximal supersymmetry* assumption (*constant* 3-form field C in their $D = 11$ supergravity interpretation). In most cases studied up to now in the physics literature, this is also the case.

In the recent articles by Konechny and Schwarz [14], however, this is probably no longer the case. For the case of $K3$ for example, which can be described as an orbifold T^4/Γ, where Γ is any discrete group, considering orbifolds corresponds to breaking *half* of the supersymmetry. Konechny and Schwarz studied the *moduli space* of *constant-curvature connections* on *finitely generated projective modules* (this should be thought of as the noncommutative analogue of fibre bundles) over algebras of the form (we follow their notation) $B^d_\theta := T^d_\theta \rtimes \mathbb{Z}_2$, where T^d_θ is our friend the noncommutative d-dimensional torus. Let us denote by $B^d := T^d \rtimes \mathbb{Z}_2$ the commutative $\mathbb{Z}_2$ toroidal orbifold (when we write T^d we mean functions on T^d to be absolutely precise, but by Gelfand's theorem these are identified). These connections correspond to $1/2$ *BPS states*. Then the volume of the moduli space is related to the number of quantum states by standard physical arguments. The first question is: does the foliation corresponding to the algebra B^d_θ have non-zero GV class? If the answer is yes, our topological discussion is of much interest; if not, one should break more supersymmetry in order to make noncommutative topological phenomena appear. More work is certainly needed in order to understand these *fractional* BPS states from the physics point of view.

Our ideas seem to be supported by two observations; the first one is made in [14]:

1. When the authors in [14] tried to count $1/4$ BPS states on the noncommutative 3-torus T^3_θ, they observed that the result agreed with the result obtained in [15] for the *commutative* 3-torus T^3. This means that the noncommutative torus alone is not enough for noncommutative topology.
2. The 0th K-theory group of the $\mathbb{Z}_2$ noncommutative toroidal orbifold B^d_θ is the *same* as the commutative $\mathbb{Z}_2$ toroidal orbifold B^d, which in turn is the same as the $\mathbb{Z}_2$-equivariant K-theory of T^d. More concretely
$$K_0(B^d_\theta) \cong K_0(B^d) := K_0(T^d \rtimes \mathbb{Z}_2) \cong K^0_{\mathbb{Z}_2}(T^d) = \mathbb{Z}^{3\cdot 2^{d-1}}.$$
The above result follows from the work of Julg [16] and Walters [17].

So to conclude, in this paper we have argued that the assumption that the compactified dimensions form a noncommutative torus will have consequences for the BH area–entropy formula, provided that the foliated torus is 'messy enough' to have resilient leaves. Our argument was purely topological.

Let us close with the following remark: in all of these articles [14] there are no *cyclic (co)homology* groups appearing, the reason *possibly* being that *topologically* these spaces are in fact commutative (tori which can be continuously deformed to the commutative case where the noncommutativity parameter θ is zero), despite the fact that they are called noncommutative. Our discussion was about foliated manifolds (tori in particular) which have indeed extra noncommutative topological charges, namely either the GV class or our new operator algebraic invariant which uses cyclic (co)homology.

Moreover, since it is very important in string theory to understand some non-supersymmetric background, it is perhaps the case that as far as noncommutative geometry is concerned, in order to have some non-trivial topological phenomena appearing (e.g. non-zero topological entropy), one must break supersymmetry completely. This suggests that an understanding of non-supersymmetric string vacua may give some better understanding of the Baum–Connes conjecture at least for the particular case of foliated manifolds and vice versa; i.e. if one wants to understand non-supersymmetric string vacua, one must use noncommutative topology. That was the second point that we tried to argue here, hoping to stimulate research in both mathematics and physics.

Finally we note that the notion of entropy also appears in the theory of C^*-algebras (see [21] and [20]). It is interesting to find a relation between topological entropy of foliated manifolds and the entropy of the C^*-algebra corresponding to the foliation.

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